Analysis of Accelerated Life Tests with Competing Failure Modes

Cornel Bunea
School of Engineering and Applied Science
George Washington University
Washington, D.C.
USA
cornel@gwu.edu

Thomas A. Mazzuchi
School of Engineering and Applied Science
George Washington University
Washington, D.C.
USA
mazzuchi@seas.gwu.edu

Abstract

Most of the accelerated life testing literature ignores the possibility of competing modes of failure. The literature that attempts to address this problem often does so by assuming independence among the competing failure modes. Rather, the failure modes often display a highly dependent structure, which is usually a function of the applied stresses. A dependent model for the competing failure modes based on a copula model for the joint failure density is presented.

1 Introduction

The presence of competing risk in Accelerated life testing (ALT) has been taken into account for some time. (McCool 1978) presents a technique for calculating estimate intervals for Weibull parameters of a primary failure mode when a secondary failure mode having the same (but unknown) Weibull shape parameter is acting. (Klein and Basu 1981,1982) presented in a series of papers, the analysis of ALT when more than one failure mode is acting. Assuming independence among competing failure modes for each stress level, the authors obtain maximum likelihood estimators when the lifetimes are exponentially or Weibull distributed, and data is type I, type II or progressively censored. A dependent competing risk model is proposed by considering a bivariate Weibull distribution as the joint survival function of two competing risks. (Nelson 1990) dedicates an entire chapter to competing failure modes in ALT. He presents graphical and analytical (maximum likelihood) methods to analyze data on a failure mode, to estimate a product life distribution when failure modes act and to estimate a product life distribution with certain failure modes eliminated. Examples of products which have multiple cause of failure are given, including insulation systems, ball bearings and industrial heaters.

In this article we develop a dependent model for the competing failure modes based on a copula model for the joint failure density. In this model we describe the relation between the applied stress and the various measures of association used to define or characterize the copula. We illustrate the improvement in inference of our model as compared to those assuming independent competing risk using actual data ((Nelson 1990), and it is collected from a temperature-accelerated life test of motor insulation).

2 Competing risks in ALT

We consider a m-constant stress level ALT. At each stress level l, $l = 1, \ldots m$, a number of n_l items are tested until a failure or a censored time occur. The failure is assumed to occur due to k competing failure modes, $X_1, X_2, \ldots X_k$. In a competing risks context, we observe the shortest of X_i, $i = 1, \ldots k$, and observe which failure mode it is. In other words we observe $Z = \min(X_1, X_2, \ldots X_k)$, $\mathbb{D} = (\delta_1, \delta_2, \ldots \delta_k)$, with

$$\delta_i = \begin{cases} 1, & if \quad X_i = \min(X_1, X_2, \ldots X_k) \\ 0, & if \quad X_i \neq \min(X_1, X_2, \ldots X_k) \end{cases}$$

For an extended overview of competing risk theory see (Crowder 2001) and (Bunea 2003).

(Nelson 1990) indicated that complete data sets are usually analyzed with standard least-squares regression analysis. Such analysis may be misleading for data with competing failure modes. The analysis should

Table 1: Motor insulation data (hours) (Nelson 1990, chapter 7) (the underlined times are censored)

$190°C$	Turn	Phase	Ground	$240°C$	Turn	Phase	Ground
1	7228	10511	$\underline{10511}$	21	1175	$\underline{1175}$	1175
2	7228	11855	$\underline{11855}$	22	$\underline{1881}$	$\underline{1881}$	1175
3	7228	11855	$\underline{11855}$	23	1521	$\underline{1881}$	$\underline{1881}$
4	8448	11855	$\underline{11855}$	24	1569	1761	$\underline{1761}$
5	9167	$\underline{12191}$	$\underline{12191}$	25	1617	$\underline{1881}$	$\underline{1881}$
6	9167	$\underline{12191}$	$\underline{12191}$	26	1665	$\underline{1881}$	$\underline{1881}$
7	9167	$\underline{12191}$	$\underline{12191}$	27	1665	$\underline{1881}$	$\underline{1881}$
8	9167	$\underline{12191}$	$\underline{12191}$	28	1713	$\underline{1881}$	$\underline{1881}$
9	10511	$\underline{12191}$	$\underline{12191}$	29	1761	$\underline{1881}$	$\underline{1881}$
10	10511	$\underline{12191}$	$\underline{12191}$	30	1953	$\underline{1953}$	$\underline{1953}$
$220°C$	Turn	Phase	Ground	$260°C$	Turn	Phase	Ground
11	1764	2436	2436	31	$\underline{1632}$	$\underline{1632}$	600
12	2436	2436	2490	32	$\underline{1632}$	$\underline{1632}$	744
13	2436	2436	2436	33	$\underline{1632}$	$\underline{1632}$	744
14	2436	$\underline{2772}$	2772	34	$\underline{1632}$	$\underline{1632}$	744
15	2436	$\underline{2436}$	2436	35	$\underline{1632}$	$\underline{1632}$	912
16	2436	$\underline{4116}$	$\underline{4116}$	36	1128	$\underline{1128}$	1128
17	3108	$\underline{4116}$	$\underline{4116}$	37	1512	$\underline{1512}$	1320
18	3108	$\underline{4116}$	$\underline{4116}$	38	1464	$\underline{1632}$	$\underline{1632}$
19	3108	3108	$\underline{3108}$	39	1608	$\underline{1608}$	1608
20	3108	$\underline{4116}$	$\underline{4116}$	40	1896	1896	1896

consist from a separate ALT model for each failure mode and a series-system model for the relationship between the failure times of each failure mode and the failure time of the component.

2.1 Motor insulation data

We consider a sample data given by (Nelson 1990), and it is collected from a temperature-accelerated life test of motor insulation. This data is a pseudo-competing risk data. The experiment was conduct in order to observe a greater number of failures for each failure mode. The motorette was kept on test and run to a second or third failure after a first failure had occurred. In actual use, the first failure from any cause ends the life of the motor. Table 1. gives the failure time for each cause (Turn, Phase, Ground). In order to keep the analysis simple we consider only to competing risk classes: Class I - Turn failures; Class II - Phase and Ground failures. Following the guide lines presented in Nelson for analysis of accelerated life testing when competing risks are present, we can obtain the estimates for each model and then a series-system model for the relationship between the failure times of each failure mode and the failure time of the component is applied. Figure 1 presents the hazard plot for Risk I and the estimates obtained by least square method. A similar picture is obtained for Risk II.

This analysis depends on the assumption of independence between risks at each stress level. Figure 2 presents the conditional subsurvival functions of the competing risks at each stress level. As indicated in (Bunea 2003), these functions seem to have an important role in model selection, via graphical interpretation. Figure 2 indicates that the conditional subsurvival function for Risk II is dominant over the conditional subsurvival function for Risk I at low stress levels, and as the stress level increases the failure mechanism is changing and the conditional subsurvival function for Risk I became dominant over the conditional subsurvival function for Risk II. This is equivalent with a random sign model in competing risk theory, but with Risks switched as the stress level increases. The random sings model is a highly dependent model, hence the

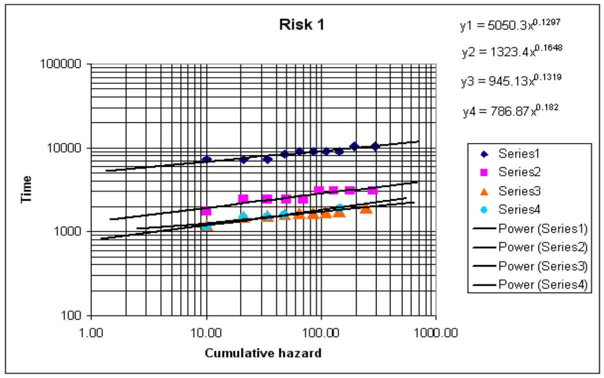

Figure 1: Hazard plot for Risk I - Turn failures

assumption of independence is not true.

3 Competing risks and copula

We assume that the dependence structure between X_1 and X_2 is given by a copula. The copula of two random variables X_1 and X_2 is the distribution C on the unit square $[0,1]^2$ of the pair $(F_{X_1}(X_1), F_{X_2}(X_2))$. The functional form of $C : [0,1]^2 \to \mathbf{R}$ is $C(u,v) \equiv H(F^{-1}_{X_1}(u), F^{-1}_{X_2}(v))$, where H is the joint distribution function of (X_1, X_2) and $F^{-1}_{X_1}$ and $F^{-1}_{X_2}$ are the right-continuous inverses of F_{X_1} and F_{X_2}. Under the assumption of independence of X_1 and X_2, the marginal distribution functions of X_1 and X_2 are uniquely determined by data. (Zheng and Klein 1995) showed the more general result that, if the copula of (X_1, X_2) is known, then the marginal distributions functions of X_1 and X_2 are uniquely determined by the competing risk data. More precisely, the marginal distributions functions F_{X_1} and F_{X_2} are solutions of the following system of ordinary differential equations:

$$\left\{ \begin{array}{l} \{1 - C_u(F_{X_1}(t), F_{X_2}(t))\}F'_{X_1}(t) = F^{*'}_{X_1}(t) \\ \{1 - C_v(F_{X_1}(t), F_{X_2}(t))\}F'_{X_2}(t) = F^{*'}_{X_2}(t) \end{array} \right.$$

with initial conditions $F_{X_1}(0) = F_{X_2}(0) = 0$, where $C_u(F_{X_1}(t), F_{X_2}(t))$ and $C_v(F_{X_1}(t), F_{X_2}(t))$ denote the first order partial derivatives $\frac{\delta}{\delta u}C(u,v)$ and $\frac{\delta}{\delta v}C(u,v)$ calculated in $(F_{X_1}(t), F_{X_2}(t))$. $F^*_{X_1}(t)$ and $F^*_{X_2}(t)$ are the subdistribution functions of X_1 and X_2.

3.1 Empirical copula

The estimators for copula density and copula are:

$$c_n(\frac{i}{n}, \frac{j}{n}) = \left\{ \begin{array}{l} \frac{1}{n} \text{ if } (x_{(i)}, y_{(j)}) \text{ is an element of the sample} \\ 0 \end{array} \right.$$

$$C_n(\frac{i}{n}, \frac{j}{n}) = \sum_{p=1}^{i} \sum_{q=1}^{j} c_n(\frac{p}{n}, \frac{q}{n}).$$

Work of (Zheng and Klein 1995) suggests that the important factor for an estimate of the marginal survival function is a reasonable guess at the strength of the association between competing risks (Kendall's τ or Spearman's ρ) and not the functional form of the copula. The estimates of these measures as a function of empirical copula are:

$$\rho = \frac{12}{n^2 - 1} \sum_{i=1}^{n} \sum_{j=1}^{n} [C_n(\frac{i}{n}, \frac{j}{n}) - \frac{i}{n}\frac{j}{n}],$$

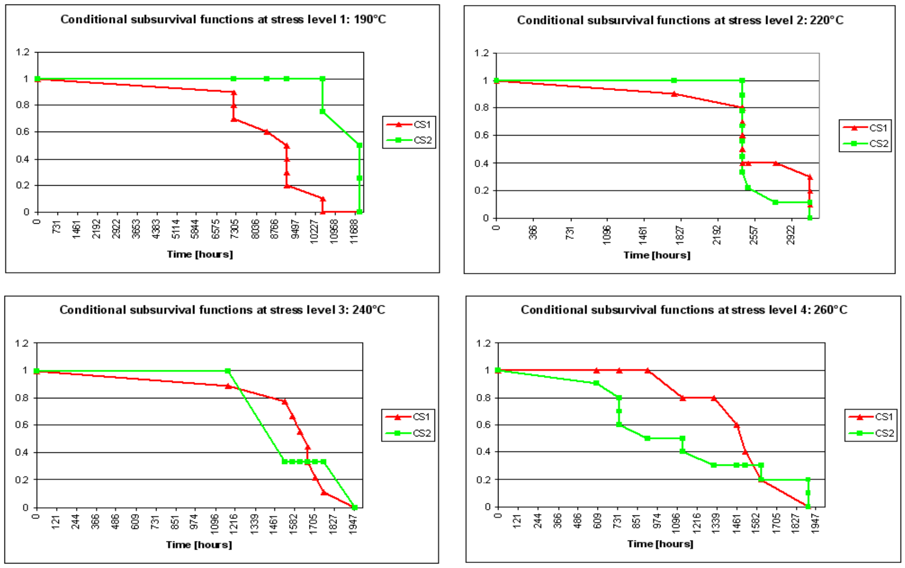

Figure 2: Conditional subsurvival functions at stress level $l = 1...4$

$$\tau = \frac{2n}{n-1}\sum_{i=2}^{n}\sum_{j=2}^{n}\sum_{p=1}^{i-1}\sum_{q=1}^{j-1} c_n(\frac{i}{n},\frac{j}{n})c_n(\frac{p}{n},\frac{q}{n}) - c_n(\frac{i}{n},\frac{q}{n})c_n(\frac{p}{n},\frac{j}{n}).$$

4 Discussion

An independent model can be misleading as simple plots of competing risk data indicate. Assuming the dependence structure between risks being specified by a copula, estimators can be found for the measure of association between competing risks (Kendall's tau or Spearman's rho). The marginal survival function can be obtained also, for each risk at stress level $l = 1...4$. If a parametric family of copula is considered (e.g. an Archimedean family), a strict relation between the measures of association and the family parameter exists. Hence, this parameter varies as the stress varies.

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